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Research Article

A semi-symmetric metric connection on an integrated contact metric structure manifold

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Abstract

In 1924, A. Friedmann and J. A. Schoten [1] introduced the idea of a semi-symmetric linear connection in a differentiable manifold. Hayden [2] has introduced the idea of metric connection with torsion in a Riemannian manifold. The properties of semi-symmetric metric connection in a Riemannian manifold have been studied by Yano [3] and others [4], [5]. The purpose of the present paper is to study some properties of semi-symmetric metric connection on an integrated contact metric structure manifold [6], several useful algebraic and geometrical properties have been studied.

Keywords: C^{∞}-manifold, integrated contact metric structure, Semi-symmetric metric connection, Riemannian connection, vector field, torsion tensor.

1. Introduction

Let M_n be a differentiable manifold of differentiability class C^{∞}. Let there exist in M_n a vector valued C^{∞} - linear function Φ, a C^{∞} - vector field η and a C^{∞} -one form ξ such that for any arbitrary vector fields X, Y, Z...... we have

(1.1) $\Phi^2(X) = a^2 X - c\xi(X)\eta$

(1.2) $\xi(\eta) = \frac{a^2}{c}$

(1.3) $\xi(\Phi(X)) = 0$

(1.4) $\Phi(\eta) = 0$

where $\Phi(X) = \bar{X}$, a is a nonzero complex number and c is an integer.

Let us agree to say that Φ gives to M_n a differentiable structure define by algebraic equation (1.1). We shall call (Φ, η, a, c, ξ) as an integrated contact structure.

Remark 1.1: The manifold M_n equipped with an integrated contact structure (Φ, η, a, c, ξ) will be called an integrated contact structure manifold.

Let there exist in M_n, a Riemannian metric G satisfying

(1.5) $G(\bar{X}, \bar{Y}) = a^2 G(X, Y) - c\xi(X)\xi(Y)$

(1.6) $G(X, \eta) = \xi(X)$

Then the C^{∞} -manifold M_n satisfying (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) is called an integrated contact metric structure manifold $(\Phi, \eta, a, c, G, \xi)$.

Remark 1.2: The integrated contact metric structure manifold $(\Phi, \eta, a, c, G, \xi)$ gives an almost norden contact metric manifold [7], Lorentzian para-contact manifold [8] or an almost Para-contact Riemannian manifold [9] according as $(a^2 = -1, c = 1), (a^2 = 1, c = -1)$ or $(a^2 = 1, c = 1)$

Agreement 1.1: An integrated contact metric structure manifold $(\Phi, \eta, a, c, G, \xi)$ will be denoted by M_n.

If we define

(1.7) $'\Phi(X, Y) = G(\bar{X}, Y) = G(X, \bar{Y})$

Where $'\Phi$ is a tensor field of the type (0, 2) then it is easy to see that

(1.8) $'\Phi(X, Y) = {}'\Phi(Y, X)$

Which shows that $'\Phi$ is symmetric in X and Y. Also we have

(1.9) $'\Phi(\bar{X}, \bar{Y}) = a^2\, {}'\Phi(X, Y)$

Theorem 1.1: In M_n, we have

(1.10)
$$(D_X\, {}'\Phi)(\bar{Y}, \bar{\bar{Z}}) + a^2 (D_X\, {}'\Phi)(\bar{\bar{Y}}, \bar{Z}) + (1 - a^2) G(\bar{\bar{Y}}, D_X \bar{\bar{Z}}) + a^2(a^2 - 1) G(\bar{Y}, D_X \bar{Z}) = 0$$

Proof: In view of (1.7), we have

$$'\Phi(Y,Z)=G(\bar{Y},Z)$$

Barring Z in the above equation and using (1.5), we get

$$'\Phi(Y,\bar{Z})=a^2 G(Y,Z)-c\xi(Y)\xi(Z)$$

Further barring Y and Z in the above equation, we get

(1.11) $$'\Phi(\bar{Y},\bar{\bar{Z}})=a^2 G(\bar{Y},\bar{Z})$$

Differentiating covariantly (1.11) along vector X and using (1.7), we get

(1.12)
$$(D_X\,'\Phi)(\bar{Y},\bar{\bar{Z}})+G(\bar{\bar{Y}},D_X\bar{\bar{Z}})-a^2G(\bar{Y},D_X\bar{Z})=0$$

Further barring Y and Z in (1.12) and using (1.1), we get

(1.13)
$$a^2(D_X\,'\Phi)(\bar{\bar{Y}},\bar{Z})+a^4G(\bar{Y},D_X\bar{Z})-a^2G(\bar{\bar{Y}},D_X\bar{\bar{Z}})=0$$

Now adding (1.12) and (1.13), we get (1.10).

2. Semi-symmetric metric connection in M_n

Agreement 2.1: Let D be the Riemannian connection. We consider a semi-symmetric metric connection B in M_n define by

(2.1) $B_XY=D_XY+\xi(Y)X-G(X,Y)\eta$

(2.1) is equivalent to

(2.2) $B_XY=D_XY+H(X,Y)$, where

(2.3) $H(X,Y)=\xi(Y)X-G(X,Y)\eta$

Let S be the torsion tensor of the connection B, then we have

(2.4) $S(X,Y)=\xi(Y)X-\xi(X)Y$

(2.5) $S(X,Y)=H(X,Y)-H(Y,X)$

Let us put

(2.6) $'H(X,Y,Z) \underset{=}{def} G(H(X,Y),Z)$

(2.7) $'S(X,Y,Z) \underset{=}{def} G(S(X,Y),Z)$

(2.8) $'S(X,Y,Z)=H(X,Y,Z)-H(Y,X,Z)$

(2.9) $'S(X,Y,Z)=\xi(Y)G(X,Z)-\xi(X)G(Y,Z)$

(2.10) $'H(X,Y,Z)=\xi(Y)G(X,Z)-\xi(Z)G(X,Y)$

Theorem 2.1: In M_n, we have

(2.11) $'H(\bar{\bar{X}},Y,Z)=a^2H(X,Y,Z)$

Proof: Barring X two times in (2.10) and using (1.1) and (2.10), we get (2.11).

Theorem 2.2: In M_n, we have

(2.12) $'S(X,Y,\bar{\bar{Z}})=a^2\,'S(X,Y,Z)$

Proof: Barring Z two times in (2.9) and using (1.1) and (2.9), we get (2.12)

Theorem 2.3: In M_n, we have

(2.13) $'H(\bar{X},Y,\bar{Z})-\,'H(\bar{X},Z,\bar{Y})=a^2\,'H(X,Y,Z)$

(2.14) $'S(\bar{X},Y,\bar{Z})-\,'S(\bar{X},Z,\bar{Y})=a^2\,'H(X,Y,Z)$

Proof: Barring X and Z in the (2.10) and using (1.3) and (1.5), we get

(2.15)
$$'H(\bar{X},Y,\bar{Z})=a^2\xi(Y)G(X,Z)-c\xi(X)\xi(Y)\xi(Z)$$

Now interchanging Y and Z in (2.10) and then barring X and Z and using (1.3) and (1.5), we get

(2.16)
$$'H(\bar{X},Z,\bar{Y})=a^2\xi(Z)G(X,Y)-c\xi(X)\xi(Y)\xi(Z)$$

Now subtracting (2.16) from (2.15) and using (2.10), we get (2.13). Further Barring X and Z in (2.9) and using (1.3) and (1.5), we get

(2.17)
$$'S(\bar{X},Y,\bar{Z})=a^2\xi(Y)G(X,Z)-c\xi(X)\xi(Y)\xi(Z)$$

Further interchanging Y and Z in (2.9) and then barring X and Y and using (1.3) and (1.5), we get

(2.18)
$$'S(\bar{X},Z,\bar{Y})=a^2\xi(Z)G(X,Y)-c\xi(X)\xi(Y)\xi(Z)$$

Now subtracting (2.18) from (2.17) and using (2.10), we get (2.14).

Theorem 2.4: In M_n, we have

(2.19) $'S(\bar{\bar{X}},Y,\bar{\bar{Z}})-\,'H(\bar{X},Y,\bar{Z})=(a^2-1)\xi(Y)G(\bar{X},\bar{Z})$

Proof: Barring X and Z in (2.17) and using (1.3), we get

(2.20) $$'S(\bar{\bar{X}},Y,\bar{\bar{Z}})=a^2\xi(Y)G(\bar{X},\bar{Z})$$

Now barring X and Z in (2.10) and using (1.3), we get

(2.21) $'H(\bar{X},Y,\bar{Z})=\xi(Y)G(\bar{X},\bar{Z})$

Subtracting (2.21) from (2.20), we get (2.19)

Theorem 2.5: In M_n, we have

(2.22) $'S(\bar{X},Y,\bar{Z})-\,'S(\bar{Y},X,\bar{Z})=a^2\,'S(X,Y,Z)$

(2.23) $'H(\bar{X},Y,\bar{Z})-\,'H(\bar{Y},X,\bar{Z})=a^2\,'S(X,Y,Z)$

Proof: From (2.17), we have

(2.24) $'S(\bar{X},Y,\bar{Z})=a^2\xi(Y)G(X,Z)-c\xi(X)\xi(Y)\xi(Z)$

Now interchanging X and Y in (2.9) and then barring Y and Z and using (1.3) and (1.5), we get

(2.25) $'S(\bar{Y},X,\bar{Z})=a^2\xi(X)G(Y,Z)-c\xi(X)\xi(Y)\xi(Z)$

Now subtracting (2.25) from (2.24) and using (2.9), we get (2.22). Again in view of (2.15), we have

(2.26) $'H(\bar{X},Y,\bar{Z})=a^2\xi(Y)G(X,Z)-c\xi(X)\xi(Y)\xi(Z)$

Now interchanging X and Y in (2.10) and then barring Y and Z and using (1.3) and (1.5), we get

(2.27) $'H(\bar{Y},X,\bar{Z}) = a^2\xi(X)G(Y,Z) - c\xi(X)\xi(Y)\xi(Z)$

Subtracting (2.27) from (2.26) and using (2.9), we get (2.23).

Theorem 2.6: In M_n, we have

(2.28) $'H(\bar{\bar{X}},Y,\bar{\bar{Z}}) + {}'H(\bar{X},Y,\bar{Z}) = (1+a^2)\xi(Y)G(\bar{X},\bar{Z})$

(2.29) $'S(\bar{X},Y,\bar{Z}) + {}'S(\bar{\bar{X}},Y,\bar{\bar{Z}}) = (1+a^2)\xi(Y)G(\bar{X},\bar{Z})$

Proof: From (2.15), we have

(2.30) $'H(\bar{X},Y,\bar{Z}) = a^2\xi(Y)G(X,Z) - c\xi(X)\xi(Y)\xi(Z)$

Barring X and Z in the above equation and using (1.3), we get

(2.31) $'H(\bar{\bar{X}},Y,\bar{\bar{Z}}) = a^2\xi(Y)G(\bar{X},\bar{Z})$

Adding (2.30) and (2.31) and using (1.1), we get (2.28). From (2.17) we have

(2.32) $'S(\bar{X},Y,\bar{Z}) = a^2\xi(Y)G(X,Z) - c\xi(X)\xi(Y)\xi(Z)$

Now barring X and Z in the above equation and using (1.3), we get

(2.33) $'S(\bar{\bar{X}},Y,\bar{\bar{Z}}) = a^2\xi(Y)G(\bar{X},\bar{Z})$

Adding (2.32) and (2.33) and using (1.5), we get (2.29).

Theorem 2.7: In M_n, we have

(2.34) $'H(\bar{\bar{X}},\bar{Z},Y) + {}'H(\bar{\bar{X}},Z,\bar{Y}) = -a^2\,{}'H(\bar{X},Y,Z)$

Proof: Interchanging Y and Z in (2.10) and then barring X, we get

(2.35) $'H(\bar{X},Z,Y) = \xi(Z)G(\bar{X},Y) - \xi(Y)G(\bar{X},Z)$

Further barring Z in (2.35) and using (1.3), we get

(2.36) $'H(\bar{X},\bar{Z},Y) = -\xi(Y)G(\bar{X},\bar{Z})$

Now barring Y in (2.35) and using (1.3), we get

(2.37) $'H(\bar{X},Z,\bar{Y}) = \xi(Z)G(\bar{X},\bar{Y})$

Adding (2.36) and (2.37) and using (1.5) and then (2.10), we get

$$'H(\bar{X},\bar{Z},Y) + {}'H(\bar{X},Z,\bar{Y}) = -a^2\,{}'H(X,Y,Z)$$

Barring X in the above equation, we get (2.34)

Theorem 2.8: In M_n, we have

(2.38) $'H(X,\bar{Y},\bar{Z}) + {}'H(\bar{X},Y,\bar{Z}) = a^2\,{}'H(X,Y,Z)$

Proof: In view of (2.10), we have

(2.39) $'H(X,Y,Z) = \xi(Y)G(X,Z) - \xi(Z)G(X,Y)$

Barring X and Y in (2.39) and using (1.3), we get

(2.40) $'H(\bar{X},\bar{Y},Z) = -\xi(Z)G(\bar{X},\bar{Y})$

Further barring X and Z in (2.39) and using (1.3), we get

(2.41) $'H(\bar{X},Y,\bar{Z}) = \xi(Y)G(\bar{X},\bar{Z})$

Adding (2.40) and (2.41) and using (1.5) and (2.10), we get (2.38)

Theorem 2.9: In M_n, we have

(2.42)
$$c\left[\xi(H(X,Y)) - \xi(H(\bar{X},\bar{Y}))\right] = (c - a^2c)\xi(X)\xi(Y) + a^2(a^2-1)G(X,Y)$$

Proof: Multiplying by ξ in (2.3), we get

(2.43) $\xi(H(X,Y)) = \xi(Y)\xi(X) - G(X,Y)\xi(\eta)$

Barring X and Y in (2.43) and using (1.3) and (1.5), we get

(2.44) $\xi(H(\bar{X},\bar{Y})) = -a^2G(X,Y)\xi(\eta) + c\xi(X)\xi(Y)\xi(\eta)$

Subtracting (2.44) from (2.43) and using (1.2), we get (2.42)

Corollary 2.1: In M_n, we have

(2.45) $'H(\bar{X},\eta,\bar{Y}) = \dfrac{a^2}{c}G(\bar{X},\bar{Y})$

(2.46) $'S(\bar{X},\eta,\bar{Y}) = \dfrac{a^2}{c}G(\bar{X},\bar{Y})$

Proof: Replacing Z by η in (2.10) and then interchanging Y and η, we get

$$'H(X,\eta,Y) = \xi(\eta)G(X,Y) - \xi(Y)G(X,\eta)$$

Now barring X and Y in the above equation and using (1.2) and (1.3), we get (2.45). Now replacing Z by η in (2.9) and then interchanging Y and η, we get

$$'S(X,\eta,Y) = \xi(\eta)G(X,Y) - \xi(X)G(\eta,Y)$$

Further barring X and Y in the above equation and using (1.2) and (1.3), we get (2.46).

Corollary 2.2: In M_n, we have

(2.47) $'H(\bar{\bar{X}},\eta,\bar{Y}) = \dfrac{a^4}{c}\,{}'F(X,Y)$

(2.48) $'S(\bar{\bar{X}},\eta,\bar{Y}) = \dfrac{a^4}{c}\,{}'F(X,Y)$

Proof: Barring X in corollary (2.45) and using (1.1), (1.6) and (1.3) we get, (2.47).

Also barring X in corollary (2.46) and using (1.1), (1.6) and (1.3) we get, (2.48).

Theorem 2.10: In a C^∞ - manifold M_n with semi-symmetric metric connection B, we have

(2.49) $B_{\bar{X}}\bar{\bar{Y}} = D_{\bar{X}}\bar{\bar{Y}} - a^2F(X,Y)\eta$

Proof: Barring X and Y in (2.1) and using (1.3) and (1.5), we get

$$B_{\bar{X}}\bar{Y} = D_{\bar{X}}\bar{Y} - a^2G(X,Y)\eta + c\xi(X)\xi(Y)\eta$$

Further barring Y in the above equation and using (1.3) and (1.7), we get (2.49).

Corollary 2.3: In M_n, we have

(2.50) $c\,(B_{\bar{X}}\xi)(\bar{\bar{Y}}) = c(D_{\bar{X}}\xi)(\bar{\bar{Y}}) - a^4G(X,\bar{Y})$

(2.51) $c\,(B_{\bar{\bar{X}}}\xi)(\bar{\bar{Y}}) = c(D_{\bar{\bar{X}}}\xi)(\bar{\bar{Y}}) - a^4G(\bar{X},\bar{Y})$

Proof: Multiplying by ξ in (2.49) and using (1.2), we get (2.50). Now barring X in (2.50) and using (1.7), we get (2.51).

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