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Research Article

A Comparative Study on Modifications of Decomposition Method

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Abstract

Many mathematical physics models are contributed to give rise to of nonlinear integral equations. In this paper, we study the performance of two recently developed modifications of well known so called Adomian's decomposition method applied using Laplace transform to nonlinear Volterra integral equations. Three nonlinear Volterra integral equations are solved analytically by implementing these modifications. From the obtained results, it may be concluded that that the modified techniques are reliable, efficient and easy to use through recursive relations that involve simple integrals. Moreover, these particular examples show the reliability and the performance of proposed modifications.

Keywords: Laplace transforms, Adomian's Decomposition Method, Taylor's series, Volterra Integral Equations.

1. Introduction

Analytical study of integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics, potential theory, electrostatics and radioactive heat transfer problems. The nonlinear Volterra integral equations arise from various natural physical and biological models such as the population dynamics, spread of epidemics, and semi-conductor devices. The essential features of these models are of wide applicable [1]. Vito Volterra thoroughly investigated the integral equation in the case when the kernel of integral is linear function. He also described a wide range of applications of integral equations with variable boundary, which is one of the most important factors in the development of the theory of integral equations. In recent years, many works have been focusing on the developing and applying of advanced and efficient methods for integral equations such as implicitly linear collocation methods [2], product integration method [3], Hermite-type collocation method [4] and analytical techniques such as Adomian's decomposition method [5,6], homotopy analysis method [7-9], homotopy perturbation method [10], the Exp-function method [11], variational iteration method [12] and the Adomian's decomposition method [13]. In this work, we investigate the performance of two modification of Adomian's decomposition method applied to non-linear Volterra integral equations of the second kind. This type of integral equations has the following form

$$u(x) = f(x) + \int_0^x k(x,t)F(u(t))dt \tag{1}$$

Eq. (1) represents a nonlinear Volterra integral equation of second kind with unknown function $u(x)$ and $F(u)$ is a non-linear function of $u(x)$, and we assumed that, the kernel $k(x,t)$ and the function $f(x)$ are analytical functions on R^2 and R, respectively.

2. Analysis of Decomposition Method

2.1 First Modification

In the first modified technique, we assume that the function can be split as follows

$$f(x) = f_1(x) + f_2(x) \tag{2}$$

Applying Laplace transform to Eq. (1), we have the recursive relation

$$u_0(s) = f_1(s),$$

$$u_1(s) = f_2(s) + L\int_0^x k(x,t)A_o(t)dt, \tag{3}$$

$$u_{n+1}(s) = L\int_0^x k(x,t)A_n(t)dt, n \geq 1.$$

Implementation the inverse Laplace transforms to (3) will produce the required solution.

$$u(x)=\sum_{n=0}^{\infty}u_n(x). \quad (4)$$

2.2 Second Modification

The main idea of the second modified technique is replacing the non-homogeneous function $f(x)$ by a series of infinite components. Ref. [14] expresses $f(x)$ in term of the Taylor series and introduces the recursive formula, after applying the Laplace transform

$$u_0(s)=f_0(s),$$
$$u_{n+1}(s)=f_{n+1}(s)+L\int_0^x k(x,t)A_n(t)dt, n\geq 0. \quad (5)$$

In Eq. (5), $f_i(s);(i=0,1,...,n)$ represents the Taylor series components of $f(s)$ and the solution can be expressed in an infinite series form as Eq. (4).

3. Numerical Applications

Example 3.1 Consider the following non-linear Volterra's integral equation

$$u(x)=\sin x+\frac{1}{8}\sin 2x-\frac{1}{4}x+\frac{1}{2}\int_0^x u^2(t)dt. \quad (6)$$

First Modification

Splitting $f(x)$ into two parts

$$f_o(x)=\sin x, f_1(x)=\frac{1}{8}\sin 2x-\frac{1}{4}x \quad (7)$$

Using the recursion relation (3), we get

$$L\{u_o(x)\}=L\{\sin x\}=\frac{1}{1+s^2}$$

$$L\{u_1(x)\}=L\{\frac{1}{8}\sin 2x-\frac{1}{4}x\}+L\{\frac{1}{2}\int_0^x u_0^2(t)dt\}=0$$

This implies that

$$L\{u_{n+1}(x)\}=L\{\int_0^x A_n(t)dt\}=0; n\geq 1 \quad (8)$$

$$U(s)=\sum_0^{\infty}U_n(s)=\frac{1}{1+s^2}+0+0+... \quad (9)$$

Taking Laplace inverse transform on both sides of the above equation

$$u(x)=\sin x \quad (10)$$

Second Modification

To apply the second modified technique, let us first expand the function $f(x)$ in terms of Taylor series expansion.

$$f(x)=x-\frac{1}{3}x^3+\frac{1}{24}x^5-\frac{17}{5040}x^7+\frac{13}{72576}x^9-... \quad (11)$$

The recursive Formula (5) gives

$$L\{u_o(x)\}=L\{x\}=\frac{1!}{s^2}$$

$$L\{u_1(x)\}=L\{-\frac{x^3}{3}\}+L\{\frac{1}{2}\int_0^x u^2{}_o(t)dt\}=-\frac{1}{s^4} \quad (12)$$

$$L\{u_2(x)\}=L\{\frac{x^5}{24}\}+L\{\frac{1}{2}\int_0^x 2u_o(t)u_1(t)dt\}=\frac{1}{s^6}$$

$$L\{u_3(x)\}=L\{-\frac{17}{5040}x^7\}+L\{\frac{1}{2}\int_0^x(2u_o(t)u_2(t)+u_1{}^2(t))dt\}=-\frac{1}{s^8}$$

$$U(s)=\sum_0^{\infty}U_n(s)=\frac{1}{s^2}-\frac{1}{s^4}+\frac{1}{s^6}-\frac{1}{s^8}+... \quad (13)$$

Taking Laplace inverse transform on both sides of the above equation

$$u(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+...=\sin x \quad (14)$$

Example 3.2 Consider the non-linear Volterra's integral equation

$$u(x)=\sec x+\tan x-\int_0^x u^2(t)dt. \quad (15)$$

First Modification

Split $f(x)$ into two parts

$$f_o(x)=\sec x, f_1(x)=\tan x. \quad (16)$$

Using the recursion relation (3), we get

$$L\{u_o(x)\}=L\{\sec x\}$$

$$L\{u_1(x)\}=L\{\tan x\}-L\{\int_0^x u^2{}_o(t)dt\}=0$$

This implies that

$$L\{u_{n+1}(x)\}=L\{\int_0^x A_n(t)dt\}=0, n\geq 1 \quad (10)$$

$$U(s)=\sum_0^{\infty}U_n(s)=L\{\sec x\}+0+0+... \quad (17)$$

Taking Laplace inverse transform on both sides of the above equation

$$u(x)=\sec x \quad (18)$$

Second Modification

To apply the second modified technique, let us first expand the function $f(x)$ in terms of Taylor series expansion.

$$f(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{5}{24}x^4+\frac{2}{15}x^5+... \quad (19)$$

According to recursive formula (5), we have

$$L\{u_o(x)\}=L\{1\}=\frac{1}{s}$$

$$L\{u_1(x)\}=L\{x\}-L\{\int_0^x u^2{}_o(t)dt\}=0$$

$$L\{u_2(x)\}=L\{\frac{1}{2}x^2\}-L\{\int_0^x 2u_o(t)u_1(t)dt\}=\frac{2!}{2s^3}$$

$$L\{u_3(x)\}=L\{\frac{1}{3}x^3\}-L\{\int_0^x(2u_o(t)u_2(t)+u_1{}^2(t))dt\}=0$$

$$L\{u_4(x)\}=L\{\frac{5}{24}x^4\}-L\{\int_0^x(2u_o(t)u_3(t)+2u_1(t)u_2(t))dt\}=\frac{5(4!)}{24s^5} \quad (20)$$

$$U(s)=\sum_0^{\infty}U_n(s)=\frac{1}{s}+0+\frac{2!}{2s^3}+0+\frac{5}{24}(\frac{4!}{s^5})+... \quad (21)$$

Taking Laplace inverse transform on both sides of the above equation

$$u(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+...=\sec x \quad (22)$$

Example 3.3 Consider the non-linear Volterra's integral equation

$$u(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xu^3(t)dt \qquad (23)$$

First Modification

Split $f(x)$ into two parts

$$f_o(x) = e^x, f_1(x) = -\frac{1}{3}xe^{3x} + \frac{1}{3}x$$

Using the modified recursion relation (3), we get

$$L\{u_o(x)\} = L\{e^x\} = \frac{1}{s-1}$$

$$L\{u_1(x)\} = L\{-\frac{1}{3}xe^{3x} + \frac{1}{3}x\} + L\{\int_0^x xu^3{}_o(t)dt\} = 0$$

This implies that

$$L\{u_{n+1}(x)\} = L\{\int_0^x A_n(t)dt\} = 0; n \ge 1 \qquad (24)$$

Second Modification

To apply the second modified technique, let us first expand the function $f(x)$ in terms of Taylor series expansion.

$$f(x) = 1 + x - \frac{1}{2}x^2 - \frac{4}{3}x^3 - \frac{35}{24}x^4 - \ldots \qquad (25)$$

According to recursive formula (5), we have

$$L\{u_o(x)\} = L\{1\} = \frac{1}{s}$$

$$L\{u_1(x)\} = L\{x\} + L\{\int_0^x xu^3{}_o(t)dt\} = \frac{1!}{s^2} + \frac{2!}{s^3}$$

$$L\{u_2(x)\} = L\{-\frac{1}{2}x^2\} + L\{\int_0^x x[3u_o^2(t)u_1(t)]dt\} = -\frac{1}{s^3} + \frac{9}{s^4} + \frac{4!}{s^5}$$

$$L\{u_3(x)\} = L\{-\frac{4}{3}x^3\} - L\{\int_0^x x[3u_o^2(t)u_2(t) + 3u_o(t)u_1^2(t)]dt\}$$
$$= -\frac{8}{s^4} + \frac{4!}{2s^5} + \frac{21(5!)}{8s^6} + \frac{6(6!)}{5s^7} \qquad (26)$$

$$U(s) = \sum_0^\infty U_n(s) = \frac{1}{s} + \frac{1!}{s^2} + \frac{2!}{s^3} - \frac{1}{s^3} + \frac{9}{s^4} + \frac{4!}{s^5} - \frac{8}{s^4} + \frac{4!}{2s^5} + \frac{21(5!)}{8s^6} + \frac{6(6!)}{5s^7} + \ldots \qquad (27)$$

Taking Laplace inverse transform on both sides of the above equation

$$u(x) = e^x. \qquad (28)$$

4. Conclusions

This paper presents the application of two modification of decomposition method for solving nonlinear Volterra's integral equations of 2nd kind. Exact solutions of the three tested problems, arising in many physical and biological models are calculated by using modifications. We note that 2nd modification minimizes the size of the calculations which produced in the first modification. In addition, it is clear that the reduction in each iteration will relieve the construction of Adomian's polynomials for the non-linear term.

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